

# THE $L_p$ MINKOWSKI PROBLEM FOR POLYTOPES FOR NEGATIVE $p$

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ABSTRACT. Existence of solutions to the  $L_p$  Minkowski problem is proved for all  $p < 0$ . For the critical case of  $p = -n$ , which is known as the centro-affine Minkowski problem, this paper contains the main result in [72] as a special case.

## 1. INTRODUCTION

A *convex body* in  $n$ -dimensional Euclidean space,  $\mathbb{R}^n$ , is a compact convex set that has non-empty interior. If  $p \in \mathbb{R}$  and  $K$  is a convex body in  $\mathbb{R}^n$  that contains the origin in its interior, then the  $L_p$  surface area measure,  $S_p(K, \cdot)$ , of  $K$  is a Borel measure on the unit sphere,  $S^{n-1}$ , defined for each Borel  $\omega \subset S^{n-1}$  by

$$S_p(K, \omega) = \int_{x \in \nu_K^{-1}(\omega)} (x \cdot \nu_K(x))^{1-p} d\mathcal{H}^{n-1}(x),$$

where  $\nu_K : \partial'K \rightarrow S^{n-1}$  is the Gauss map of  $K$ , defined on  $\partial'K$ , the set of boundary points of  $K$  that have a unique outer unit normal, and  $\mathcal{H}^{n-1}$  is  $(n-1)$ -dimensional Hausdorff measure.

The  $L_p$  surface area measure was introduced by Lutwak [41]. The  $L_p$  surface area measure contains three important measures as special cases: the  $L_1$  surface area measure is the classic surface area measure; the  $L_0$  surface area measure is the cone-volume measure; the  $L_{-n}$  surface area measure is the centro-affine surface area measure. Today, the  $L_p$  surface area measure is a central notation in convex geometry analysis, and appeared in, e.g., [3, 8, 21–28, 37–52, 54, 56–60, 65–67].

The following  $L_p$  Minkowski problem that posed by Lutwak [41] is considered as one of the most important problems in modern convex geometry analysis.

**$L_p$  Minkowski problem:** *Find necessary and sufficient conditions on a finite Borel measure  $\mu$  on  $S^{n-1}$  so that  $\mu$  is the  $L_p$  surface area measure of a convex body in  $\mathbb{R}^n$ .*

The associated partial differential equation for the  $L_p$  Minkowski problem is the following Mong-Ampère type equation: For a given positive function  $f$  on the unit sphere, solve

$$(1.1) \quad h^{1-p} \det(h_{ij} + h\delta_{ij}) = f,$$

where  $h_{ij}$  is the covariant derivative of  $h$  with respect to an orthonormal frame on  $S^{n-1}$  and  $\delta_{ij}$  is the Kronecker delta.

The solutions of the  $L_p$  Minkowski problem have important applications to affine isoperimetric inequalities, see, e.g., Zhang [70], Lutwak, Yang and Zhang [46], Ciachi, Lutwak,

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Yang and Zhang [12], Haberl and Schuster [25–27]. The solutions to the  $L_p$  Minkowski problem are also related with some important flows (see, e.g., [1, 2, 61, 62]).

When  $p = 1$ , the  $L_p$  Minkowski problem is the classical Minkowski problem. The existence and uniqueness for the solution of this problem was solved by Minkowski, Aleksandrov, and Fenchel and Jessen (see Schneider [57] for references). Regularity of the Minkowski problem was studied by e.g., Caffarelli [7], Cheng and Yau [10], Nirenberg [53] and Pogorelov [55].

For  $p \neq 1$ , the  $L_p$  Minkowski problem was studied by, e.g., Lutwak [41], Lutwak and Oliker [42], Lutwak, Yang and Zhang [47], Chou and Wang [11], Guan and Lin [19], Hug, Lutwak, Yang and Zhang [22], Böröczky, Hegedűs and Zhu [4], Böröczky, Lutwak, Yang and Zhang [5, 6], Chen [9], Dou and Zhu [14], Haberl, Lutwak, Yang and Zhang [22], Huang, Liu and Xu [30], Jian, Lu and Wang [32], Jian and Wang [34], Jiang, Wang and Wei [35], Lu and Wang [36], Stancu [61, 62], Sun and Long [63] and Zhu [71–73]. Analogues of the Minkowski problems were studied in, e.g., [13, 15, 16, 18, 20, 29, 68].

The uniqueness of solutions to the  $L_p$  Minkowski for  $p > 1$  can be shown by applying the  $L_p$  Minkowski inequality established by Lutwak [41]. However, little is known about the  $L_p$  Minkowski inequality for the case where  $p < 1$ . This is one of the main reasons that most of the previous work on the  $L_p$  Minkowski problem was limited to the case where  $p > 1$ .

The critical case where  $p = -n$  of the  $L_p$  Minkowski problem is called the centro-affine Minkowski problem, which describes the centro-affine surface area measure. This problem is especially important due to the affine invariance of the partial differential equation (1.1). It is known that the centro-affine Minkowski problem has connections with several important geometric problems (see, e.g., Jian and Wang [34] for reference). The centro-affine Minkowski problem was explicitly posed by Chow and Wang [11]. Recently, the centro-affine Minkowski problem was studied by Lu and Wang [36] for rotationally symmetric case and was studied by Zhu [72] for discrete measures.

When  $p < -n$ , very few results are known for the  $L_p$  Minkowski problem. So far as the author knows, in  $\mathbb{R}^2$ , the  $L_p$  Minkowski problem for all  $p < 0$  was studied by Dou and Zhu [14], Sun and Long [63]. It is the aim of this paper to study the  $L_p$  Minkowski problem for all  $p < 0$  and  $n \geq 2$ .

It is known that the Minkowski problem and the  $L_p$  Minkowski problem (for  $p > 1$ ) for arbitrary measures can be solved by an approximation argument by first solving the polytopal case (see, e.g., [31] or [57] pp. 392–393). This is one of the reasons why the Minkowski problem and the  $L_p$  Minkowski problem for polytopes are of great importance.

A *polytope* in  $\mathbb{R}^n$  is the convex hull of a finite set of points in  $\mathbb{R}^n$  provided that it has positive  $n$ -dimensional volume. The convex hull of a subset of these points is called a *facet* of the polytope if it lies entirely on the boundary of the polytope and has positive  $(n - 1)$ -dimensional volume. Let  $P$  be a polytope which contains the origin in its interior with  $N$  facets whose outer unit normals are  $u_1, \dots, u_N$ , and such that the facet with outer unit normal  $u_k$  has area  $a_k$  and distance  $h_k$  from the origin for all  $k \in \{1, \dots, N\}$ . Then,

$$S_p(P, \cdot) = \sum_{k=1}^N h_k^{1-p} a_k \delta_{u_k}(\cdot).$$

where  $\delta_{u_k}$  denotes the delta measure that is concentrated at the point  $u_k$ .

A finite subset  $U$  of  $S^{n-1}$  is said to be *in general position* if any  $k$  elements of  $U$ ,  $1 \leq k \leq n$ , are linearly independent.

In [72], the author solved the centro-affine Minkowski problem for polytopes whose outer unit normals are in general position:

**Theorem A.** *Let  $\mu$  be a discrete measure on the unit sphere  $S^{n-1}$ . Then  $\mu$  is the centro-affine surface area measure of a polytope whose outer unit normals are in general position if and only if the support of  $\mu$  is in general position and not concentrated on a closed hemisphere.*

A linear subspace  $X$  ( $0 < \dim X < n$ ) of  $\mathbb{R}^n$  is said to be *essential* with respect to a Borel measure  $\mu$  on  $S^{n-1}$  if  $X \cap \text{supp}(\mu)$  is not concentrated on any closed hemisphere of  $X \cap S^{n-1}$ .

Obviously, if the support of a discrete measure  $\mu$  is in general position, then the set of essential subspaces of  $\mu$  is empty. On the other hand, in  $\mathbb{R}^n$  ( $n \geq 3$ ), one can easily construct a discrete measure  $\mu$  such that  $\mu$  does not have essential subspace but the support of  $\mu$  is not in general position. Therefore, the set of discrete measures whose supports are in general position is a subset of the set of discrete measures that do not have essential subspaces.

It is the aim of this paper to solve the  $L_p$  Minkowski problem for discrete measures that do not have essential subspaces. Obviously, the following main theorem of this paper contains Theorem A as a special case.

**Theorem 1.1.** *Let  $p < 0$  and  $\mu$  be a discrete measure on the unit sphere  $S^{n-1}$ . Then  $\mu$  is the  $L_p$  surface area measure of a polytope whose  $L_p$  surface area measure does not have essential subspace if and only if  $\mu$  does not have essential subspace and not concentrated on a closed hemisphere.*

## 2. PRELIMINARIES

In this section, we standardize some notations and list some basic facts about convex bodies. For general references regarding convex bodies, see, e.g., [17, 57, 64].

The sets in this paper are subsets of the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . For  $x, y \in \mathbb{R}^n$ , we write  $x \cdot y$  for the standard inner product of  $x$  and  $y$ ,  $|x|$  for the Euclidean norm of  $x$ , and  $S^{n-1}$  for the unit sphere of  $\mathbb{R}^n$ .

Suppose  $S$  is a subset of  $\mathbb{R}^n$ , then the positive hull,  $\text{pos}(S)$ , of  $S$  is the set of all positive combinations of any finitely many elements of  $S$ . Let  $\text{lin}(S)$  be the smallest linear subspace of  $\mathbb{R}^n$  containing  $S$ . The diameter of a subset,  $S$ , of  $\mathbb{R}^n$  is defined by

$$d(S) = \max\{|x - y| : x, y \in S\}.$$

The convex hull of a subset,  $S$ , of  $\mathbb{R}^n$  is defined by

$$\text{Conv}(S) = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1 \text{ and } x, y \in S\}.$$

For convex bodies  $K_1, K_2$  in  $\mathbb{R}^n$  and  $s_1, s_2 \geq 0$ , the Minkowski combination is defined by

$$s_1 K_1 + s_2 K_2 = \{s_1 x_1 + s_2 x_2 : x_1 \in K_1, x_2 \in K_2\}.$$

The *support function*  $h_K : \mathbb{R}^n \rightarrow \mathbb{R}$  of a convex body  $K$  is defined, for  $x \in \mathbb{R}^n$ , by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Obviously, for  $s \geq 0$  and  $x \in \mathbb{R}^n$ ,

$$h(sK, x) = h(K, sx) = sh(K, x).$$

If  $K$  is a convex body in  $\mathbb{R}^n$  and  $u \in S^{n-1}$ , then the *support set*  $F(K, u)$  of  $K$  in direction  $u$  is defined by

$$F(K, u) = K \cap \{x \in \mathbb{R}^n : x \cdot u = h(K, u)\}.$$

The *Hausdorff distance* of two convex bodies  $K_1, K_2$  in  $\mathbb{R}^n$  is defined by

$$\delta(K_1, K_2) = \inf\{t \geq 0 : K_1 \subset K_2 + tB^n, K_2 \subset K_1 + tB^n\},$$

where  $B^n$  is the unit ball.

Let  $\mathcal{P}$  be the set of polytopes in  $\mathbb{R}^n$ . If the unit vectors  $u_1, \dots, u_N$  are not concentrated on a closed hemisphere, let  $\mathcal{P}(u_1, \dots, u_N)$  be the subset of  $\mathcal{P}$  such that a polytope  $P \in \mathcal{P}(u_1, \dots, u_N)$  if the set of the outer unit normals of  $P$  is a subset of  $\{u_1, \dots, u_N\}$ . Let  $\mathcal{P}_N(u_1, \dots, u_N)$  be the subset of  $\mathcal{P}(u_1, \dots, u_N)$  such that a polytope  $P \in \mathcal{P}_N(u_1, \dots, u_N)$  if,  $P \in \mathcal{P}(u_1, \dots, u_N)$ , and  $P$  has exactly  $N$  facets.

### 3. AN EXTREMAL PROBLEM RELATED TO THE $L_p$ MINKOWSKI PROBLEM

Suppose  $p < 0$ ,  $\alpha_1, \dots, \alpha_N > 0$ , the unit vectors  $u_1, \dots, u_N$  are not concentrated on a closed hemisphere, and  $P \in \mathcal{P}(u_1, \dots, u_N)$ . Define the function,  $\Phi_P : \text{Int}(P) \rightarrow \mathbb{R}$ , by

$$\Phi_P(\xi) = \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi \cdot u_k)^p.$$

In this section, we study the extremal problem

$$(3.1) \quad \sup\left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\}.$$

The main purpose of this section is to prove that a dilation of the solution to problem (3.1) solves the corresponding  $L_p$  Minkowski problem.

**Lemma 3.1.** *If  $p < 0$ ,  $\alpha_1, \dots, \alpha_N > 0$ , the unit vectors  $u_1, \dots, u_N$  are not concentrated on a closed hemisphere and  $P \in \mathcal{P}(u_1, \dots, u_N)$ , then there exists a unique  $\xi(P) \in \text{Int}(P)$  such that*

$$\Phi_P(\xi(P)) = \inf_{\xi \in \text{Int}(P)} \Phi_P(\xi).$$

*Proof.* Since  $p < 0$ , the function  $f(t) = t^p$  is strictly convex on  $(0, +\infty)$ . Hence, for  $0 < \lambda < 1$  and  $\xi_1, \xi_2 \in \text{Int}(P)$ ,

$$\begin{aligned} \lambda \Phi_P(\xi_1) + (1 - \lambda) \Phi_P(\xi_2) &= \lambda \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi_1 \cdot u_k)^p + (1 - \lambda) \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi_2 \cdot u_k)^p \\ &= \sum_{k=1}^N \alpha_k [\lambda (h(P, u_k) - \xi_1 \cdot u_k)^p + (1 - \lambda) (h(P, u_k) - \xi_2 \cdot u_k)^p] \\ &\geq \sum_{k=1}^N \alpha_k [h(P, u_k) - (\lambda \xi_1 + (1 - \lambda) \xi_2) \cdot u_k]^p \\ &= \Phi_P(\lambda \xi_1 + (1 - \lambda) \xi_2). \end{aligned}$$

Equality hold if and only if  $\xi_1 \cdot u_k = \xi_2 \cdot u_k$  for all  $k = 1, \dots, N$ . Since  $u_1, \dots, u_N$  are not concentrated on a closed hemisphere,  $\mathbb{R}^n = \text{lin}\{u_1, \dots, u_N\}$ . Thus,  $\xi_1 = \xi_2$ . Hence,  $\Phi_P$  is strictly convex on  $\text{Int}(P)$ .

From the fact that  $P \in \mathcal{P}(u_1, \dots, u_N)$ , we have, for any  $x \in \partial P$ , there exists a  $u_{i_0} \in \{u_1, \dots, u_N\}$  such that

$$h(P, u_{i_0}) = x \cdot u_{i_0}.$$

Thus,  $\Phi_P(\xi) \rightarrow \infty$  whenever  $\xi \in \text{Int}(P)$  and  $\xi \rightarrow x$ . Therefore, there exists a unique interior point  $\xi(P)$  of  $P$  such that

$$\Phi_P(\xi(P)) = \inf_{\xi \in \text{Int}(P)} \Phi_P(\xi).$$

□

Obviously, for  $\lambda > 0$  and  $P \in \mathcal{P}(u_1, \dots, u_N)$ ,

$$(3.2) \quad \xi(\lambda P) = \lambda \xi(P),$$

and if  $P_i \in \mathcal{P}(u_1, \dots, u_N)$  and  $P_i$  converges to a polytope  $P$ , then  $P \in \mathcal{P}(u_1, \dots, u_N)$ .

**Lemma 3.2.** *If  $p < 0$ ,  $\alpha_1, \dots, \alpha_N > 0$ , the unit vectors  $u_1, \dots, u_N$  are not contained in a closed hemisphere,  $P_i \in \mathcal{P}(u_1, \dots, u_N)$ , and  $P_i$  converges to a polytope  $P$ , then  $\lim_{i \rightarrow \infty} \xi(P_i) = \xi(P)$  and*

$$\lim_{i \rightarrow \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

*Proof.* Since  $P_i$  converges to  $P$  and  $\xi(P_i) \in \text{Int}(P_i)$ ,  $\xi(P_i)$  is bounded. Let  $\xi_0$  be the limit point of a subsequence,  $\xi(P_{i_j})$ , of  $\xi(P_i)$ . We claim that  $\xi_0 \in \text{Int}(P)$ . Otherwise,  $\xi_0$  is a boundary point of  $P$  with  $\lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi_{P_{i_j}}) = \infty$ , which contradicts the fact that

$$(3.3) \quad \overline{\lim}_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \overline{\lim}_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi(P)) = \Phi(\xi(P)) < \infty.$$

We claim that  $\xi_0 = \xi(P)$ . Otherwise,

$$\begin{aligned} \lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi(P_{i_j})) &= \Phi_P(\xi_0) \\ &> \Phi_P(\xi(P)) \\ &= \lim_{j \rightarrow \infty} \Phi_{P_{i_j}}(\xi(P)). \end{aligned}$$

This contradicts the fact that

$$\Phi_{P_{i_j}}(\xi(P_{i_j})) \leq \Phi_{P_{i_j}}(\xi(P)).$$

Hence,  $\lim_{i \rightarrow \infty} \xi(P_i) = \xi(P)$  and

$$\lim_{i \rightarrow \infty} \Phi_{P_i}(\xi(P_i)) = \Phi_P(\xi(P)).$$

□

**Lemma 3.3.** *If  $p < 0$ ,  $\alpha_1, \dots, \alpha_N > 0$ , the unit vectors  $u_1, \dots, u_N$  are not concentrated on a closed hemisphere and  $P \in \mathcal{P}(u_1, \dots, u_N)$ , then*

$$\sum_{k=1}^N \alpha_k \frac{u_k}{[h(P, u_k) - \xi(P) \cdot u_k]^{1-p}} = 0.$$

*Proof.* Define  $f : \text{Int}(P) \rightarrow \mathbb{R}^n$  by

$$f(x) = \sum_{k=1}^N \alpha_k (h(P, u_k) - x \cdot u_k)^p.$$

By conditions,

$$f(\xi(P)) = \inf_{x \in \text{Int}(P)} f(x).$$

Thus,

$$\sum_{k=1}^N \alpha_k \frac{u_{k,i}}{[h(P, u_k) - \xi(P) \cdot u_k]^{1-p}} = 0,$$

for all  $i = 1, \dots, n$ , where  $u_k = (u_{k,1}, \dots, u_{k,n})^T$ . Therefore,

$$\sum_{k=1}^N \alpha_k \frac{u_k}{[h(P, u_k) - \xi(P) \cdot u_k]^{1-p}} = 0.$$

□

**Lemma 3.4.** *Suppose  $p < 0$ ,  $\alpha_1, \dots, \alpha_N > 0$ , the unit vectors  $u_1, \dots, u_N$  are not concentrated on a closed hemisphere, and there exists a  $P \in \mathcal{P}_N(u_1, \dots, u_N)$  with  $\xi(P) = o$ ,  $V(P) = 1$  such that*

$$\Phi_P(o) = \sup \left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\}.$$

Then,

$$S_p(P_0, \cdot) = \sum_{k=1}^N \alpha_k \delta_{u_k}(\cdot),$$

where  $P_0 = \left( \sum_{j=1}^N \alpha_j h(P, u_j)^p / n \right)^{\frac{1}{n-p}} P$ .

*Proof.* By conditions, there exists a polytope  $P \in \mathcal{P}_N(u_1, \dots, u_N)$  with  $\xi(P) = o$  and  $V(P) = 1$  such that

$$\Phi_P(o) = \sup \left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\},$$

where  $\Phi_Q(\xi) = \sum_{k=1}^N \alpha_k (h(Q, u_k) - \xi \cdot u_k)^p$ .

For  $\tau_1, \dots, \tau_N \in \mathbb{R}$ , choose  $|t|$  small enough so that the polytope  $P_t$  defined by

$$P_t = \bigcap_{i=1}^N \{x : x \cdot u_i \leq h(P, u_i) + t\tau_i\}$$

has exactly  $N$  facets. By [57] (Lemma 7.5.3),

$$\frac{\partial V(P_t)}{\partial t} = \sum_{i=1}^N \tau_i a_i,$$

where  $a_i$  is the area of  $F(P, u_i)$ . Let  $\lambda(t) = V(P_t)^{-\frac{1}{n}}$ , then  $\lambda(t)P_t \in \mathcal{P}_N^n(u_1, \dots, u_N)$ ,  $V(\lambda(t)P_t) = 1$  and

$$(3.4) \quad \lambda'(0) = -\frac{1}{n} \sum_{i=1}^N \tau_i S_i.$$

Define  $\xi(t) := \xi(\lambda(t)P_t)$ , and

$$(3.5) \quad \begin{aligned} \Phi(t) &:= \min_{\xi \in \lambda(t)P_t} \sum_{k=1}^N \alpha_k (\lambda(t)h(P_t, u_k) - \xi \cdot u_k)^p \\ &= \sum_{k=1}^N \alpha_k (\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k)^p. \end{aligned}$$

It follows from Lemma 3.3 that

$$\sum_{k=1}^N \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - \xi(t) \cdot u_k]^{1-p}} = 0,$$

for  $i = 1, \dots, n$ , where  $u_k = (u_{k,1}, \dots, u_{k,n})^T$ . In addition, since  $\xi(P)$  is the origin,

$$(3.6) \quad \sum_{k=1}^N \alpha_k \frac{u_k}{h(P, u_k)^{1-p}} = 0.$$

Let  $F = (F_1, \dots, F_n)$  be a function from an open neighbourhood of the origin in  $\mathbb{R}^{n+1}$  to  $\mathbb{R}^n$  such that

$$F_i(t, \xi_1, \dots, \xi_n) = \sum_{k=1}^N \alpha_k \frac{u_{k,i}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})]^{1-p}}$$

for  $i = 1, \dots, n$ . Then,

$$\left. \frac{\partial F_i}{\partial t} \right|_{(t, \xi_1, \dots, \xi_n)} = \sum_{k=1}^N \frac{(p-1)\alpha_k u_{k,i} [\lambda'(t)h(P_t, u_k) + \lambda(t)\tau_k]}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})]^{2-p}},$$

$$\left. \frac{\partial F_i}{\partial \xi_j} \right|_{(t, \xi_1, \dots, \xi_n)} = \sum_{k=1}^N \frac{(1-p)\alpha_k u_{k,i} u_{k,j}}{[\lambda(t)h(P_t, u_k) - (\xi_1 u_{k,1} + \dots + \xi_n u_{k,n})]^{2-p}}$$

are continuous on a small neighbourhood of  $(0, 0, \dots, 0)$  with

$$\left( \left. \frac{\partial F}{\partial \xi} \right|_{(0, \dots, 0)} \right)_{n \times n} = \sum_{k=1}^N \frac{(1-p)\alpha_k}{h(P, u_k)^{2-p}} u_k \cdot u_k^T,$$

where  $u_k u_k^T$  is an  $n \times n$  matrix.

Since  $u_1, \dots, u_N$  are not contained in a closed hemisphere,  $\mathbb{R}^n = \text{lin}\{u_1, \dots, u_N\}$ . Thus, for any  $x \in \mathbb{R}^n$  with  $x \neq 0$ , there exists a  $u_{i_0} \in \{u_1, \dots, u_N\}$  such that  $u_{i_0} \cdot x \neq 0$ . Then,

$$\begin{aligned} x^T \cdot \left( \sum_{k=1}^N \frac{(1-p)\alpha_k}{h(P, u_k)^{2-p}} u_k \cdot u_k^T \right) \cdot x &= \sum_{k=1}^N \frac{(1-p)\alpha_k}{h(P, u_k)^{2-p}} (x \cdot u_k)^2 \\ &\geq \frac{(1-p)\alpha_{i_0}}{h(P, u_{i_0})^{2-p}} (x \cdot u_{i_0})^2 > 0. \end{aligned}$$

Therefore,  $(\frac{\partial F}{\partial \xi}|_{(0, \dots, 0)})$  is positive defined. By this, the fact that  $F_i(0, \dots, 0) = 0$  for all  $i = 1, \dots, n$ , the fact that  $\frac{\partial F_i}{\partial \xi_j}$  is continuous on a neighbourhood of  $(0, 0, \dots, 0)$  for all  $0 \leq i, j \leq n$  and the implicit function theorem, we have

$$\xi'(0) = (\xi'_1(0), \dots, \xi'_n(0))$$

exists.

From the fact that  $\Phi(0)$  is an extreme value of  $\Phi(t)$  (in Equation (3.5)), Equation (3.4) and Equation (3.6), we have

$$\begin{aligned} 0 &= \Phi'(0)/p \\ &= \sum_{k=1}^N \alpha_k h(P, u_k)^{p-1} (\lambda'(0)h(P, u_k) + \tau_k - \xi'(0) \cdot u_k) \\ &= \sum_{k=1}^N \alpha_k h(P, u_k)^{p-1} \left[ -\frac{1}{n} \left( \sum_{i=1}^N a_i \tau_i \right) h(P, u_k) + \tau_k \right] - \xi'(0) \cdot \left[ \sum_{k=1}^N \alpha_k \frac{u_k}{h(P, u_k)^{1-p}} \right] \\ &= \sum_{k=1}^N \alpha_k h(P, u_k)^{p-1} \tau_k - \left( \sum_{i=1}^N a_i \tau_i \right) \frac{\sum_{k=1}^N \alpha_k h(P, u_k)^p}{n} \\ &= \sum_{k=1}^N \left( \alpha_k h(P, u_k)^{p-1} - \frac{\sum_{j=1}^N \alpha_j h(P, u_j)^p}{n} a_k \right) \tau_k. \end{aligned}$$

Since  $\tau_1, \dots, \tau_N$  are arbitrary,

$$\frac{\sum_{j=1}^N \alpha_j h(P, u_j)^p}{n} h(P, u_k)^{1-p} a_k = \alpha_k,$$

for all  $k = 1, \dots, N$ . By letting

$$P_0 = \left( \frac{\sum_{j=1}^N \alpha_j h(P, u_j)^p}{n} \right)^{\frac{1}{n-p}} P,$$

we have

$$S_p(P_0, \cdot) = \sum_{k=1}^N \alpha_k \delta_{u_k}(\cdot).$$

□



## 4. THE PROOF OF THE MAIN THEOREM

In this section, we prove the main theorem of this paper.

The following lemmas will be needed.

**Lemma 4.1.** *Let  $\{h_{1j}\}_{j=1}^\infty, \dots, \{h_{Nj}\}_{j=1}^\infty$  be  $N$  ( $N \geq 2$ ) sequences of real numbers. Then, there exists a subsequence,  $\{j_n\}_{n=1}^\infty$ , of  $\mathbb{N}$  and a rearrangement,  $i_1, \dots, i_N$ , of  $1, \dots, N$  such that*

$$h_{i_1 j_n} \leq h_{i_2 j_n} \leq \dots \leq h_{i_N j_n},$$

for all  $n \in \mathbb{N}$ .

*Proof.* For each fixed  $j$ , the number of the possible order (from small to big) of  $h_{1j}, \dots, h_{Nj}$  is  $N!$ . Therefore, there exists a subsequence,  $\{j_n\}_{n=1}^\infty$ , of  $\mathbb{N}$  and a rearrangement,  $i_1, \dots, i_N$ , of  $1, \dots, N$  such that

$$h_{i_1 j_n} \leq h_{i_2 j_n} \leq \dots \leq h_{i_N j_n},$$

for all  $n \in \mathbb{N}$ . □

**Lemma 4.2.** *Suppose the unit vectors  $u_1, \dots, u_N$  are not concentrated on a closed hemisphere, and for any subspace,  $X$ , of  $\mathbb{R}^n$  with  $1 \leq \dim X \leq n-1$ ,  $\{u_1, \dots, u_N\} \cap X$  is concentrated on a closed hemisphere of  $S^{n-1} \cap X$ . If  $P_m$  is a sequence of polytopes with  $V(P_m) = 1$ ,  $o \in \text{Int}(P_m)$  and  $P_m \in \mathcal{P}(u_1, \dots, u_N)$ , then  $P_m$  is bounded.*

*Proof.* We only need to prove that if the diameter,  $d(P_i)$ , of  $P_i$  is not bounded, then there exists a subspace,  $X$ , of  $\mathbb{R}^n$  with  $1 \leq \dim(X) \leq n-1$  and  $\{u_1, \dots, u_N\} \cap X$  is not concentrated on a closed hemisphere of  $S^{n-1} \cap X$ .

Let  $\mu$  be a discrete measure on the unit sphere such that  $\text{supp}(\mu) = \{u_1, \dots, u_N\}$ ,  $\mu(u_i) = \alpha_i > 0$  for  $1 \leq i \leq N$ . Obviously, we only need to prove the lemma under the condition that  $\xi(P_m) = o$  for all  $m \in \mathbb{N}$ .

By Lemma 4.1, we may assume that

$$(4.0) \quad h(P_m, u_1) \leq \dots \leq h(P_m, u_N).$$

By this and the condition that  $V(P_m) = 1$  and  $\lim_{m \rightarrow \infty} d(P_m) = \infty$ ,

$$\lim_{m \rightarrow \infty} h(P_m, u_1) = 0 \text{ and } \lim_{m \rightarrow \infty} h(P_m, u_N) = \infty.$$

By this and (4.0), there exists an  $i_0$  ( $1 \leq i_0 \leq N$ ) such that

$$(4.1) \quad \overline{\lim}_{m \rightarrow \infty} \frac{h(P_m, u_{i_0})}{h(P_m, u_1)} = \infty,$$

and for  $1 \leq i \leq i_0 - 1$

$$(4.2) \quad \overline{\lim}_{m \rightarrow \infty} \frac{h(P_m, u_i)}{h(P_m, u_1)}$$

exists and equals to a positive number.

Let

$$\Sigma = \text{pos}\{u_1, \dots, u_{i_0-1}\}$$

and

$$\Sigma^* = \{x \in \mathbb{R}^n : x \cdot u_i \leq 0 \text{ for all } 1 \leq i \leq i_0 - 1\}.$$

Let  $1 \leq j \leq i_0 - 1$  and  $x \in \Sigma^* \cap S^{n-1}$ . From the condition that  $\xi(P_m)$  is the origin and Lemma 3.3, we have

$$\sum_{i=0}^N \frac{\alpha_i(x \cdot u_i)}{[h(P_m, u_i)]^{1-p}} = 0.$$

By this and the fact that  $x \in \Sigma^* \cap S^{n-1}$ ,

$$\begin{aligned} 0 &\geq \alpha_j(x \cdot u_j) \\ &= - \sum_{i \neq j} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i(x \cdot u_i) \\ &\geq \sum_{i \geq i_0} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i(x \cdot u_i) \\ &\geq - \sum_{i \geq i_0} \left[ \frac{h(P_m, u_j)}{h(P_m, u_i)} \right]^{1-p} \alpha_i \end{aligned}$$

By this, (4.0), (4.1) and (4.2),  $\alpha_j(x \cdot u_j)$  is no bigger than 0 and no less than any negative number. Hence,

$$x \cdot u_j = 0$$

for all  $j = 1, \dots, i_0 - 1$  and  $x \in \Sigma^* \cap S^{n-1}$ . Thus,

$$(4.3) \quad \Sigma^* \cap \text{lin}\{u_1, \dots, u_{i_0-1}\} = \{0\}.$$

Obviously,  $\{u_1, \dots, u_{i_0-1}\}$  is not concentrated on a closed hemisphere of  $S^{n-1} \cap \text{lin}\{u_1, \dots, u_{i_0-1}\}$ . Otherwise, there exists an  $x_0 \in \text{lin}\{u_1, \dots, u_{i_0-1}\}$  with  $x_0 \neq 0$  such that  $x_0 \cdot u_i \leq 0$  for all  $1 \leq i \leq i_0 - 1$ . This contradicts with (4.3).

We next prove that

$$\text{lin}\{u_1, \dots, u_{i_0-1}\} \neq \mathbb{R}^n.$$

Otherwise, from the fact that  $u_1, \dots, u_{i_0-1}$  are not concentrated on a closed hemisphere of

$$\text{lin}\{u_1, \dots, u_{i_0-1}\} \cap S^{n-1},$$

we have, the convex hull of  $\{u_1, \dots, u_{i_0-1}\}$  (denoted by  $Q$ ) is a polytope in  $\mathbb{R}^n$  and contains the origin as an interior. Let  $F$  be a facet of  $Q$  such that  $\{su_{i_0} : s > 0\} \cap F \neq \emptyset$ . Since  $F$  is the union of finite  $(n-1)$ -dimensional simplexes and the vertexes of these simplexes are subsets of  $\{u_1, \dots, u_{i_0-1}\}$ , there exists a subset,  $\{u_{i_1}, \dots, u_{i_n}\}$ , of  $\{u_1, \dots, u_{i_0-1}\}$  such that

$$u_{i_0} \in \text{pos}\{u_{i_1}, \dots, u_{i_n}\}.$$

Since  $o \in \text{Int}(Q)$ , there exists  $r > 0$  such that  $rB^n \subset Q$ . Choose  $t > 0$  such that  $tu \in F \cap \text{pos}\{u_{i_1}, \dots, u_{i_n}\}$ . Then,

$$tu = \beta_{i_1}u_{i_1} + \dots + \beta_{i_n}u_{i_n},$$

where  $\beta_{i_1}, \dots, \beta_{i_n} \geq 0$  with  $\beta_{i_1} + \dots + \beta_{i_n} = 1$ . If we let  $a_{i_j} = \beta_{i_j}/t$  for  $j = 1, \dots, n$ , we have

$$u = a_{i_1}u_{i_1} + \dots + a_{i_n}u_{i_n}.$$

Obviously,  $a_{i_j} \geq 0$  with

$$a_{i_j} = \beta_{i_j}/t \leq 1/r$$

for all  $j = 1, \dots, n$ . Hence,

$$\begin{aligned} h(P_m, u_{i_0}) &= h(P_m, a_{i_1} u_{i_1} + \dots + a_{i_n} u_{i_n}) \\ &\leq a_{i_1} h(P_m, u_{i_1}) + \dots + a_{i_n} h(P_m, u_{i_n}) \\ &\leq \frac{1}{r} [h(P_m, u_{i_1}) + \dots + h(P_m, u_{i_n})], \end{aligned}$$

for all  $m \in \mathbb{N}$ . This contradicts (4.1) and (4.2). Therefore,

$$\text{lin}\{u_1, \dots, u_{i_0-1}\} \neq \mathbb{R}^n.$$

Let  $X = \text{lin}\{u_1, \dots, u_{i_0-1}\}$ . Then,  $1 \leq \dim X \leq n-1$  but  $\{u_1, \dots, u_N\} \cap X = \{u_1, \dots, u_{i_0-1}\}$  is not concentrated on a closed hemisphere of  $S^{n-1} \cap X$ , which contradicts the conditions of this lemma. Therefore,  $d(P_m)$  is bounded.  $\square$

The following lemmas will be needed (see, e.g., [73]).

**Lemma 4.3.** *If  $P$  is a polytope in  $\mathbb{R}^n$  and  $v_0 \in S^{n-1}$  with  $V_{n-1}(F(P, v_0)) = 0$ , then there exists a  $\delta_0 > 0$  such that for  $0 \leq \delta < \delta_0$*

$$V(P \cap \{x : x \cdot v_0 \geq h(P, v_0) - \delta\}) = c_n \delta^n + \dots + c_2 \delta^2,$$

where  $c_n, \dots, c_2$  are constants that depend on  $P$  and  $v_0$ .

**Lemma 4.4.** *Suppose  $p < 0$ ,  $\alpha_1, \dots, \alpha_N > 0$ , and the unit vectors  $u_1, \dots, u_N$  are not concentrated on a hemisphere. If for any subspace  $X$  with  $1 \leq \dim X \leq n-1$ ,  $\{u_1, \dots, u_N\} \cap X$  is always concentrated on a closed hemisphere of  $S^{n-1} \cap X$ , then there exists a  $P \in \mathcal{P}_N(u_1, \dots, u_N)$  such that  $\xi(P) = o$ ,  $V(P) = 1$ , and*

$$\Phi_P(o) = \sup\left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\},$$

where  $\Phi_Q(\xi) = \sum_{k=1}^N \alpha_k (h(Q, u_k) - \xi \cdot u_k)^p$ .

*Proof.* Obviously, for  $P, Q \in \mathcal{P}(u_1, \dots, u_N)$ , if there exists a  $x \in \mathbb{R}^n$  such that  $P = Q + x$ , then

$$\Phi_P(\xi(P)) = \Phi_Q(\xi(Q)).$$

Thus, we can choose a sequence of polytopes  $P_i \in \mathcal{P}(u_1, \dots, u_N)$  with  $\xi(P_i) = o$  and  $V(P_i) = 1$  such that  $\Phi_{P_i}(o)$  converges to

$$\sup\left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\}.$$

By the conditions of this lemma and Lemma 4.2,  $P_i$  is bounded. From the Blaschke selection theorem, there exists a subsequence of  $P_i$  that converges to a polytope  $P$  such that  $P \in \mathcal{P}(u_1, \dots, u_N)$ ,  $V(P) = 1$ ,  $\xi(P) = o$  and

$$(4.4) \quad \Phi_P(o) = \sup\left\{ \inf_{\xi \in \text{Int}(Q)} \Phi_Q(\xi) : Q \in \mathcal{P}(u_1, \dots, u_N) \text{ and } V(Q) = 1 \right\}.$$

We claim that  $F(P, u_i)$  are facets for all  $i = 1, \dots, N$ . Otherwise, there exists an  $i_0 \in \{1, \dots, N\}$  such that

$$F(P, u_{i_0})$$

is not a facet of  $P$ .

Choose  $\delta > 0$  small enough so that the polytope

$$P_\delta = P \cap \{x : x \cdot u_{i_0} \leq h(P, u_{i_0}) - \delta\} \in \mathcal{P}(u_1, \dots, u_N),$$

and (by Lemma 4.3)

$$V(P_\delta) = 1 - (c_n \delta^n + \dots + c_2 \delta^2),$$

where  $c_n, \dots, c_2$  are constants that depend on  $P$  and direction  $u_{i_0}$ .

From Lemma 3.2, for any  $\delta_i \rightarrow 0$  it always true that  $\xi(P_{\delta_i}) \rightarrow o$ . We have,

$$\lim_{\delta \rightarrow 0} \xi(P_\delta) = o.$$

Let  $\delta$  be small enough so that  $h(P, u_k) > \xi(P_\delta) \cdot u_k + \delta$  for all  $k \in \{1, \dots, N\}$ , and let

$$\lambda = V(P_\delta)^{-\frac{1}{n}} = (1 - (c_n \delta^n + \dots + c_2 \delta^2))^{-\frac{1}{n}}.$$

From this and Equation (3.2), we have

(4.5)

$$\begin{aligned} \Phi_{\lambda P_\delta}(\xi(\lambda P_\delta)) &= \sum_{k=1}^N \alpha_k (h(\lambda P_\delta, u_k) - \xi(\lambda P_\delta) \cdot u_k)^p \\ &= \lambda^p \sum_{k=1}^N \alpha_k (h(P_\delta, u_k) - \xi(P_\delta) \cdot u_k)^p \\ &= \lambda^p \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p - \alpha_{i_0} \lambda^p (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})^p \\ &\quad + \alpha_{i_0} \lambda^p (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta)^p \\ &= \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p + (\lambda^p - 1) \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p \\ &\quad + \alpha_{i_0} \lambda^p \left[ (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta)^p - (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})^p \right] \\ &= \Phi_P(\xi(P_\delta)) + B(\delta), \end{aligned}$$

where

$$\begin{aligned} B(\delta) &= (\lambda^p - 1) \left( \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p \right) \\ &\quad + \alpha_{i_0} \lambda^p \left[ (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta)^p - (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})^p \right] \\ &= \left[ (1 - (c_n \delta^n + \dots + c_2 \delta^2))^{-\frac{p}{n}} - 1 \right] \left( \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p \right) \\ &\quad + \alpha_{i_0} \lambda^p \left[ (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta)^p - (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})^p \right]. \end{aligned}$$

From the facts that  $d_0 = d(P) > h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} > h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta > 0$ ,  $p < 0$  and the fact that  $f(t) = t^p$  is convex on  $(0, \infty)$ , we have

$$(h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta)^p - (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})^p > (d_0 - \delta)^p - d_0^p > 0.$$

Hence,

$$\begin{aligned}
(4.6) \quad B(\delta) &= (\lambda^p - 1) \left( \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p \right) \\
&\quad + \alpha_{i_0} \lambda^p \left[ (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0} - \delta)^p - (h(P, u_{i_0}) - \xi(P_\delta) \cdot u_{i_0})^p \right] \\
&> \left[ (1 - (c_n \delta^n + \dots + c_2 \delta^2))^{-\frac{p}{n}} - 1 \right] \left( \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p \right) \\
&\quad + \alpha_{i_0} \lambda^p [(d_0 - \delta)^p - d_0^p].
\end{aligned}$$

On the other hand,

$$(4.7) \quad \lim_{\delta \rightarrow 0} \sum_{k=1}^N \alpha_k (h(P, u_k) - \xi(P_\delta) \cdot u_k)^p = \sum_{k=1}^N \alpha_k h(P, u_k)^p,$$

$$(4.8) \quad (d_0 - \delta)^p - d_0^p > 0,$$

and

$$\begin{aligned}
(4.9) \quad &\lim_{\delta \rightarrow 0} \frac{(1 - (c_n \delta^n + \dots + c_2 \delta^2))^{-\frac{p}{n}} - 1}{(d_0 - \delta)^p - d_0^p} \\
&= \lim_{\delta \rightarrow 0} \frac{(-\frac{p}{n})(1 - (c_n \delta^n + \dots + c_2 \delta^2))^{-\frac{p}{n}-1} (-n c_n \delta^{n-1} - \dots - 2 c_2 \delta)}{p(d_0 - \delta)^{p-1}(-1)} = 0.
\end{aligned}$$

From Equations (4.6), (4.7), (4.8), (4.9), and the fact that  $p < 0$ , we have  $B(\delta) > 0$  for small enough  $\delta > 0$ . From this and Equation (4.5), there exists a  $\delta_0 > 0$  such that  $P_{\delta_0} \in \mathcal{P}(u_1, \dots, u_N)$  and

$$\Phi_{\lambda_0 P_{\delta_0}}(\xi(\lambda_0 P_{\delta_0})) > \Phi_P(\xi(P_{\delta_0})) \geq \Phi_P(\xi(P)) = \Phi_P(o),$$

where  $\lambda_0 = V(P_{\delta_0})^{-\frac{1}{n}}$ . Let  $P_0 = \lambda_0 P_{\delta_0} - \xi(\lambda_0 P_{\delta_0})$ , then  $P_0 \in \mathcal{P}^n(u_1, \dots, u_N)$ ,  $V(P_0) = 1$ ,  $\xi(P_0) = o$  and

$$(4.10) \quad \Phi_{P_0}(o) < \Phi_P(o).$$

This contradicts Equation (4.4). Therefore,  $P \in \mathcal{P}_N(u_1, \dots, u_N)$ .  $\square$

Now we have prepared enough to prove the main theorem of this paper. We only need to prove the following:

**Theorem 4.5.** *Suppose  $p < 0$ ,  $\alpha_1, \dots, \alpha_N > 0$ , and the unit vectors  $u_1, \dots, u_N$  are not concentrated on a hemisphere. If for any subspace  $X$  with  $1 \leq \dim X \leq n-1$ ,  $\{u_1, \dots, u_N\} \cap X$  is always concentrated on a closed hemisphere of  $S^{n-1} \cap X$ , then there exists a polytope  $P_0 \in \mathcal{P}_N(u_1, \dots, u_N)$  such that*

$$S_p(P_0, \cdot) = \sum_{k=1}^N \alpha_k \delta_{u_k}(\cdot).$$

*Proof.* Theorem 4.5 can be directly got by Lemma 3.4 and Lemma 4.4.  $\square$

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